Abrupt Changes in the Multipole Moments of a Gravitating Body

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Abstract

An example is described in which an asymptotically flat static vacuum Weyl space-time experiences a sudden change across a null hypersurface in the multipole moments of its isolated axially symmetric source. A light–like shell and an impulsive gravitational wave are identified, both having the null hypersurface as history. The stress–energy in the shell is dominated (at large distance from the source) by the jump in the monopole moment (the mass) of the source with the jump in the dipole moment mainly responsible for the stress being anisotropic. The gravitational wave owes its existence principally to the jump in the quadrupole moment of the source confirming what would be expected. This serves as a model of a cataclysmic astrophysical event such as a supernova.

1 Introduction

Very few examples are known describing gravitational waves emitted from an isolated source having its wave-fronts homeomorphic to a 2-dimensional sphere ("spherical radiation"). The principal example is the Robinson-Trautman family of solutions [1]. These are special, however, because if the wave fronts are sufficiently smooth (no conical singularities) and the Riemann tensor has no line singularity, then these solutions approach Schwarzschild's solution exponentially in time [2]. A limiting case is Penrose's spherical impulsive wave propagating through flat space-time [3]. More recent examples in the present context have also been found by Alekseev and Griffiths [4]. When these solutions have the property that the Riemann tensor of the space-time containing the history of the wave involves a Dirac delta function which is singular on the null hypersurface history of the wave, then the coefficient of the delta function is singular along a generator of the null hypersurface. This is known as a wire singularity (or line singularity). The object of this contribution is to present a rare example of a wire singularity-free impulsive gravitational wave in vacuum which is asymptotically spherical [5]. To construct it, we use the Barrabès-Israel formalism for light-like shells [7].

2 Weyl static solutions in the Bondi form

We consider the asymptotically flat Weyl static axially symmetric solutions [8] of Einstein's vacuum field equations in the Bondi form [5], [9].

$$ds^2 = -r^2 \left\{ f^{-1} d\theta^2 + f \, \sin^2\theta \, d\phi^2 \right\} + 2g \, du \, dr + 2h \, du \, d\theta + c \, du^2 \ , \quad (1)$$

with

$$f = 1 - \frac{Q}{r^3} \sin^2\theta + O\left(r^{-4}\right) \ , \quad g = 1 + O\left(r^{-4}\right) \quad (2)$$

$$h = \frac{2D}{r} \sin\theta + \frac{3Q}{r^2} \sin\theta \, \cos\theta + O\left(r^{-3}\right) \ , \quad (3)$$

$$c = 1 - \frac{2m}{r} - \frac{2D}{r^2} \cos\theta - \frac{Q}{r^3} \left(3\cos^2\theta - 1\right) + O\left(r^{-4}\right) \ . \quad (4)$$

where the parameters (m, D, Q) can be identified respectively as the mass, the dipole moment and the quadripole moment of the isolated source [5], [9]. In the form (1), the hypersurfaces $u = const.$ are exactly null [5] (for all r and not just for large r). Neglecting $O\left(r^{-4}\right)$-terms, the hypersurfaces $u = const.$ are generated by the integral lines of the futur-pointing null vector $\partial/\partial r$ and r is an affine parameter along them. Moreover, these hypersurfaces are asymptotically future null cones [5].

We subdivide the space–time M with line–element (1) into two halves M^- and M^+ having $u = 0$ as common boundary. To the past of $u = 0$, corresponding to $u < 0$, the space–time M^- is endowed with the metric (1) with parameters $\{m_-, D_-, Q_-, \ldots\}$, metric coefficients $\{f, g, h, c\}$ given by (2)-(4) and coordinates $x^\mu_- = (\theta, \phi, r, u)$, while to the future

of $u = 0$, corresponding to $u > 0$, the space–time M^+ is given by (1), metric coefficients $\{f_+, g_+, h_+, c_+\}$ given by (2)-(4) with parameters $\{m_+, D_+, Q_+, \ldots\}$ and coordinates $x^\mu_+ = (\theta_+, \phi_+, r_+, u)$. For convenience, we have taken $u_+ = u$ and use (θ, ϕ, r) as intrinsic coordinates on $u = 0$. We now re-attach M^- and M^+ on $u = 0$ requiring only the continuity of the metric. From the perspective of the space–time $M^- \cup M^+$, we have an asymptotically flat Weyl solution M^- undergoing an abrupt finite jump in its multipole moments across a null hypersurface $u = 0$ resulting in the Weyl solution M^+. Applying now the Barrabès-Israel formalism [7], we study the physical properties of the null hypersurface $u = 0$. We find that it is the history of a light-like shell. In the system of coordinates of the M^- side, the non-vanishing components of the stress-energy tensor of the shell are [5]

$$
\begin{aligned}
S^{11} &= O\left(r^{-7}\right) , \qquad S^{22} = O\left(r^{-7}\right) , \\
16\pi\, S^{13} &= -\frac{3\,[D]}{r^4}\sin\theta + O\left(r^{-5}\right) , \\
16\pi\, S^{33} &= -\frac{4[m]}{r^2} - \frac{12\,[D]}{r^3}\cos\theta + \frac{3\,[Q]}{r^4}\,(5 - 11\,\cos^2\theta) + O\left(r^{-5}\right)
\end{aligned}
\tag{5}
$$

where the square brackets denote the jump across $u = 0$ of the quantity contained therein. The surface energy density of the shell measured by a radially moving observer is [5] [7] (up to a positive constant factor)

$$
\sigma \;:=\; -\frac{1}{4\pi r^2}\left\{[m] + \frac{3\,[D]}{r}\cos\theta - \frac{3\,[Q]}{4r^2}\,(5 - 11\,\cos^2\theta) + O\left(r^{-3}\right)\right\} \tag{6}
$$

It is natural to assume $[m] < 0$ so the source suffers a loss of mass. The results obtained in ref. [7] showed that the impulsive part of the Weyl tensor of the space-time $M^+ \cup M^-$ splits into two parts which can be identified as a matter part and a wave part. This splitting has been carried out explicitly in ref. [5], demonstrating that in general, a light-like shell is accompanied by an impulsive gravitational wave which propagate independently along the null hypersurface $u = 0$. Introducing a null tetrad asymptotically parallel transported along the integral curves of $\partial/\partial r$, we find that the non-vanishing Newmann-Penrose components of the matter part of the delta function in the Weyl tensor are [5]

$$
{}^M\Psi_3 \;=\; \frac{3\sqrt{2}\,[D]}{4\,r^3}\sin\theta + O\left(r^{-4}\right) , \quad {}^M\Psi_2 = O\left(r^{-5}\right) \tag{7}
$$

It is predominantly type III (with n^μ as degenerate principal null direction) in the Petrov classification because of the anisotropy in the stress which in turn is due to $[D] \neq 0$. The leading term of ${}^M\Psi_3$ has clearly no singularity for $0 \le \theta \le \pi$ and thus *no wire singularity.* The only non-vanishing Newmann-Penrose component of the wave part is [5]

$$
{}^W\Psi_4 = -\frac{3\,[Q]}{4r^4}\,(3 - 7\,\cos^2\theta) + O\left(r^{-5}\right) . \tag{8}
$$

This impulsive gravitational wave is Petrov type N (with n^{μ} as four-fold degenerate principal null direction) and clearly owes its existence primarily to the jump in the quadrupole moment of the source across $u = 0$ and also is manifestly *free of wire singularities.* Finally, analysing the induced metric on $u = 0$, we see that, for a given r (which is asymptotically an affine parameter along the null geodesics of $u = 0$), the corresponding 2-surfaces are topologically spherical when neglecting $O\left(r^{-4}\right)$-terms. Hence the light–like shell and the impulsive gravitational wave can be considered asymptotically spherical in this sense. We think that this example is a model of a cataclysmic astrophysical event such as a supernova which is likely to produce a burst of neutrinos travelling outward with the speed of light (modelled by the light–like shell) accompanied by a burst of outgoing gravitational waves (modelled by the impulsive wave). Another example of the coexistence of wire singularity-free spherical light-like shell and impulsive gravitational wave has been obtained by the same authors using a Kerr space-time undergoing a sudden change in the magnitude and in the direction of the angular momentum of the black-hole [6].

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